

Deligne's conjecture on the Lefschetz trace formula for p^n -torsion étale cohomology

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March 3, 2013

Abstract

Deligne's conjecture is the Lefschetz trace formula for correspondences defined over a finite field. In this paper, we prove an analogous statement of Deligne's conjecture with respect to p^n -torsion étale cohomology under certain conditions, where p is the characteristic of the base field.

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1 Introduction

In abstract algebraic geometry, we have the Lefschetz trace formula for correspondences of proper smooth schemes, which is a generalization of the Lefschetz fixed-point formula. The trace formula does not hold when schemes are not proper or smooth. However, in the positive characteristic case, we have a nice Lefschetz trace formula for correspondences twisted by a sufficiently large power of the Frobenius endomorphism. It was conjectured by Deligne, and proved by Fujiwara [Fu] under the most general situation. This trace formula is an important tool in arithmetic geometry. For example, it appears in the proof of the local Langlands correspondence for GL_n over a p -adic field by Harris-Taylor [HT], and the Langlands correspondence for GL_n over a function field by Lafforgue [La].

We first recall Deligne's conjecture. We fix a prime number p and take a prime number $\ell \neq p$. Let q be a power of p , \mathbb{F}_q the finite field with q elements and k an algebraic closure of \mathbb{F}_q . Let X and Γ be separated k -schemes of finite type and $a: \Gamma \rightarrow X \times X$ a morphism of k -schemes. We assume that X , Γ and a are defined over \mathbb{F}_q . We put $a_1 = \text{pr}_1 \circ a$ and $a_2 = \text{pr}_2 \circ a$, where pr_1 (resp. pr_2) is the first (resp. second) projection of $X \times_k X$. We denote by Fr_X the q -th power Frobenius endomorphism on X . We write $a^{(m)}$ for the correspondence such that $a_1^{(m)} = \text{Fr}_X^m \circ a_1$ and $a_2^{(m)} = a_2$. We write $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ for the derived category of bounded complexes of $\overline{\mathbb{Q}}_\ell$ -sheaves with constructible cohomology (for more detail of this category, see for example [Ek]).

Theorem 1.1 (Deligne's conjecture). *We assume that a_1 is proper and a_2 is quasi-finite. Then there exists an integer N depending only on X , Γ and a such that, for any integer $m \geq N$, any object $K \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ and any morphism $u \in \text{Hom}(a_1^{(m)*} K, a_2^! K)$, the equation*

$$\text{Tr}(u_! \mid R\Gamma_c(X, K)) = \sum_{z \in \text{Fix } a^{(m)}} \text{naive-loc}_z(u)$$

holds.

In the above equation, $u_!$ is defined by the composition

$$R\Gamma_c(X, K) \xrightarrow{\text{adj}} R\Gamma_c(\Gamma, a_1^{(m)*} K) \xrightarrow{u} R\Gamma_c(\Gamma, a_2^! K) \xrightarrow{\text{adj}} R\Gamma_c(X, K)$$

and $\text{naive-loc}_z(u)$ is defined by the trace of the composition

$$K_w = (a_1^{(m)*} K)_z \xrightarrow{u_z} (a_2^! K)_z \hookrightarrow \bigoplus_{z' \in a_2^{-1}(w)} (a_2^! K)_{z'} = (a_2^! a_2^! K)_w \xrightarrow{\text{adj}} K_w,$$

where we put $w = a_1^{(m)}(z) = a_2(z)$.

In this paper, we will show an analogous statement to Theorem 1.1 for étale cohomology of p^n -torsion sheaves. Let X_0 and Γ_0 be separated \mathbb{F}_q -schemes of finite type and $a_0: \Gamma_0 \rightarrow X_0 \times_{\mathbb{F}_q} X_0$ an \mathbb{F}_q -morphism. We write Φ_{X_0} for the p -th power map on the structure sheaf of X_0 . Let G_0 be a constructible \mathbb{Z}/p^n -sheaf on X_0 .

Theorem 1.2. *We assume a_1 is proper and a_2 is étale. Then there exists an integer N depending on X_0, Γ_0, a_0 and G_0 such that, for any integer $m \geq N$ and any $u_0 \in \text{Hom}(a_{01}^{(m)*} G_0, a_{02}^* G_0)$, we have*

$$\text{Tr}(u_! \mid R\Gamma_c(X, G)) = \sum_{z \in \text{Fix } a^{(m)}} \text{Tr}(u_z \mid G_z)$$

under one of the following conditions:

(i) (Corollary 3.2)

- $n = 1$,
- $a_2 = \text{id}_X$ and a_1 is an automorphism of finite order,
- $\text{ord } u = \text{ord } a_1$.

(ii) (Theorem 6.1)

- $n = 1$,
- a_0 is a restriction of a correspondence $\bar{a}_0: \bar{\Gamma}_0 \rightarrow \bar{X}_0 \times_{\mathbb{F}_q} \bar{X}_0$ such that \bar{X} and $\bar{\Gamma}$ are proper smooth k -schemes, X_0 (resp. Γ_0) is an open \mathbb{F}_q -subscheme of \bar{X}_0 (resp. $\bar{\Gamma}_0$), \bar{a} is a closed immersion, \bar{a}_2 is étale and $\bar{X} \setminus X$ is a Cartier divisor,
- G_0 is the pull-back of a smooth constructible \mathbb{F}_p -sheaf on \bar{X}_0 .

(iii) (Theorem 7.1)

- \bar{a} is a closed immersion,
- \bar{X} and $\bar{\Gamma}$ are proper smooth k -schemes,
- there exists a lift $(\mathcal{X}_0, \tilde{\Gamma}_0, \tilde{a}_0, \Phi_{\mathcal{X}_0})$ of $(X_0, \Gamma_0, a_0, \Phi_{X_0})$ to $W_n(\mathbb{F}_q)$,
- $H^i(X, G)$ (resp. $H^i(\mathcal{X}, G \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{\mathcal{X}})$) is free over \mathbb{Z}/p^n (resp. $W_n(k)$) for each i .

In the case (i) and (ii), we can take $N = 1$.

The first case for $a_1 = \text{id}$ is already known ([SGA4 $\frac{1}{2}$, Fonction L mod. ℓ^n]). By using the method in [DL], we extend the result to the case where a_1 is an automorphism of finite order.

For the second and last cases, we generalize the proof of the trace formula for the Frobenius correspondence. The outline of the proof is as follows. First, by tensoring with a structure sheaf, we transform the left-hand side of the formula in Theorem 1.2 into a trace of an endomorphism of cohomology groups with respect to coherent sheaves. After that, we apply certain trace formula called “Woods Hole formula” to this term. Then the term transforms into a sum of traces of stalks of a homomorphism of coherent sheaves at fixed points. Finally, by a calculation of the stalks, we have the result.

We devote Section 2 to recalling the terminology of cohomological operations and correspondences. In Section 3 we prove Theorem 1.2 in the case (i). In

Section 4 we prove the Woods Hole formula which is a Lefschetz trace formula for coherent sheaves. In Section 5 we recall some linear algebra to prove the cases (ii) and (iii). Under the preliminaries above, we prove Theorem 1.2 in the case (ii) (resp. the case (iii)) in Section 6 (resp. Section 7). Finally, we show an example such that the trace formula holds, an application to elliptic curves, and a counterexample for p -adic coefficients in Section 8.

Notation

- In this paper, we fix a prime number p . We denote by \mathbb{F}_q the finite field of q elements where q is a power of p and fix an algebraic closure k of \mathbb{F}_q . For any field K of characteristic p , we write $W_n(K)$ for the ring of Witt vectors of length n over K .
- For any $W_n(\mathbb{F}_q)$ -object \mathcal{X}_0 , we write $\mathcal{X} = \mathcal{X}_0 \otimes_{W_n(\mathbb{F}_q)} W_n(k)$ for the base change of \mathcal{X}_0 by $W_n(\mathbb{F}_q) \rightarrow W_n(k)$.
- Let X_0 be a \mathbb{F}_q -scheme. We write $\text{Fr}_{X_0}: X_0 \rightarrow X_0$ for the Frobenius endomorphism on X_0 , which is the pair of the identity on the underlying space of X_0 and the q -th power map on the structure sheaf of X_0 . We write Fr_X for the base change of Fr_{X_0} by $\mathbb{F}_q \rightarrow k$.
- Let $f_0: X_0 \rightarrow Y_0$ and $a_0: \Gamma_0 \rightarrow X_0 \times_{\mathbb{F}_q} X_0$ be morphisms of \mathbb{F}_q -schemes. We put $f_0^{(m)} = \text{Fr}_{Y_0}^m \circ f_0 = f_0 \circ \text{Fr}_{X_0}^m$ and $a_0^{(m)} = (a_{01}^{(m)}, a_{02})$, where we write a_{01} (resp. a_{02}) for the composition of a_0 and the first (resp. second) projection of $X_0 \times_{\mathbb{F}_q} X_0$. We also define $f^{(m)}$ and $a^{(m)}$ in the same way.
- Let A be a ring. We denote by $D(A)$ the derived category of the category of A -modules. We denote by $D_{\text{perf}}(A)$ the full subcategory of $D(A)$ which consists of perfect complexes, that is, complexes which are quasi-isomorphic to bounded complexes of projective A -modules of finite type.
- Let X be a scheme and \mathcal{A} a sheaf of rings on X . We denote by $D(X, \mathcal{A})$ the derived category of the category of \mathcal{A} -module sheaves on X .
- We denote by $\otimes_{\mathcal{A}}^L: D^-(X, \mathcal{A}) \times D^-(X, \mathcal{A}) \rightarrow D^-(X, \mathcal{A})$ the derived tensor product functor. If \mathcal{F} is a flat \mathcal{A} -module, we write $\otimes_{\mathcal{A}} \mathcal{F}$ instead of $\otimes_{\mathcal{A}}^L \mathcal{F}$.
- We denote by $D_{\text{perf}}(X, \mathcal{A})$ the full subcategory of $D(X, \mathcal{A})$ which consists of perfect complexes, that is, complexes which are locally quasi-isomorphic to bounded complexes of locally free \mathcal{A} -modules of finite type.
- If \mathcal{A} is a constant sheaf Λ whose value is a noetherian ring, we denote by $D_{\text{ctf}}^b(X, \Lambda)$ the category of complexes of finite tor-dimension with constructible cohomology (see for example [SGA4 $\frac{1}{2}$, Rapport]).
- If $\mathcal{A} = \mathcal{O}_X$, we simply write $D(\mathcal{O}_X)$ (resp. $D_{\text{perf}}(\mathcal{O}_X)$) for $D(X, \mathcal{O}_X)$ (resp. $D_{\text{perf}}(X, \mathcal{O}_X)$).

Acknowledgment

It is a great pleasure for the author to thank Professor Yoichi Mieda. He suggested many things about the problem, checked the preliminary manuscripts carefully, and pointed out some mistakes in it.

The author is obliged to professor Yuichiro Taguchi for inviting the author warmly into his office and giving valuable comments for this paper.

2 Cohomological correspondences

In the section, we recall notation of cohomological operations and correspondences.

2.1 Cohomological operations

2.1.1 Sheaves of Λ -modules

Let Λ be a noetherian torsion ring and S a spectrum of an artinian local ring. Let $f: X \rightarrow Y$ be a morphism of separated schemes of finite type over S .

We denote by $\mathcal{C}(X, \Lambda)$ (resp. $\mathcal{C}(Y, \Lambda)$) the category of étale sheaves of Λ -modules on X (resp. Y). We denote the direct (resp. inverse) image functor associated to f by $f_*: \mathcal{C}(X, \Lambda) \rightarrow \mathcal{C}(Y, \Lambda)$ (resp. by $f^*: \mathcal{C}(Y, \Lambda) \rightarrow \mathcal{C}(X, \Lambda)$). Since f^* is an exact functor, it induces a functor from $D(Y, \Lambda)$ to $D(X, \Lambda)$ which we also denote by f^* . Since f_* is a left exact functor and f is finite-dimensional, it induces a right derived functor $Rf_*: D(X, \Lambda) \rightarrow D(Y, \Lambda)$. We often denote Rf_* by f_* for simplicity. When f is the structure morphism, we denote Rf_* by $R\Gamma(X, \bullet)$.

For any complex $K \in D(X, \Lambda)$, we write $\text{adj}: K \rightarrow f_*f^*K$ for the adjunction map.

Let $j: X \hookrightarrow \overline{X}$ be an open immersion. We write $j_!: \mathcal{C}(X, \Lambda) \rightarrow \mathcal{C}(\overline{X}, \Lambda)$ for the extension-by-zero functor. Since $j_!$ is exact, it induces a functor from $D(X, \Lambda)$ to $D(\overline{X}, \Lambda)$ which we also denote by $j_!$.

By Nagata's compactification theorem, we can decompose $f: X \rightarrow Y$ as follows:

$$\begin{array}{ccc} X & \xrightarrow{j} & \overline{X} \\ & \searrow f & \downarrow \overline{f} \\ & & Y, \end{array}$$

where \overline{f} is a proper morphism and j is an open immersion. We define the direct image functor with proper support $f_!: D(X, \Lambda) \rightarrow D(Y, \Lambda)$ by the composite of $j_!$ and \overline{f}_* . When f is the structure morphism of X , we denote $f_!$ by $R\Gamma_c(X, \bullet)$.

The functor f^* sends $D_{\text{ctf}}^b(Y, \Lambda)$ to $D_{\text{ctf}}^b(X, \Lambda)$. On the other hand, by [SGA4, Exp. XIV, Thm. 1.1 and Exp. XVII, Thm. 5.2.10], the functor $f_!$ sends $D_{\text{ctf}}^b(X, \Lambda)$ to $D_{\text{ctf}}^b(Y, \Lambda)$. In particular, if we take K in $D_{\text{ctf}}^b(X, \Lambda)$ then $R\Gamma_c(X, K)$ belongs to $D_{\text{perf}}(\Lambda)$.

We assume that $g: Y \rightarrow X$ is finite and étale. Then, for any abelian étale sheaf G on X , we can define a morphism

$$\mathrm{Tr}_{g,G}: g_*g^*G \rightarrow G$$

such that it is compatible with étale base change and if Y is a disjoint sum of d copies of X then $\mathrm{Tr}: g_*g^*G = G^d \rightarrow G$ is the morphism $(x_i) \mapsto \sum_i x_i$, where x_1, \dots, x_d are sections of G (see for example [SGA4, Exp. VII, Sect. 5]). Since the functor g_*g^* is exact, we can extend it to a morphism of complexes $\mathrm{Tr}_{g,K}: g_*g^*K \rightarrow K$. We note that the morphism $\mathrm{Tr}_{g,K}$ is functorial with respect to K .

We remove the assumption that g is finite. We write $g = \bar{g} \circ j$ for a decomposition of g , where $j: Y \hookrightarrow \bar{Y}$ is an open immersion and $\bar{g}: \bar{Y} \rightarrow X$ is a proper morphism. Then \bar{g} is finite étale morphism, hence we can define a morphism

$$\mathrm{Tr}_{g,K}: g_*g^*K \rightarrow K$$

by the composition

$$g_*g^*K = \bar{g}_*j_*j^*g^*K \xrightarrow{\mathrm{adj}} \bar{g}_*\bar{g}^*K \xrightarrow{\mathrm{Tr}_{\bar{g},K}} K.$$

2.1.2 Sheaves of \mathcal{O}_X -modules

For a scheme X , we denote by $\mathcal{C}(\mathcal{O}_X)$ the category of \mathcal{O}_X -sheaves. Let $f: X \rightarrow Y$ be a morphism of separated schemes of finite type over S .

We can naturally define the direct image functor $f_*: \mathcal{C}(\mathcal{O}_X) \rightarrow \mathcal{C}(\mathcal{O}_Y)$ and its right derived functor from $D(\mathcal{O}_X)$ to $D(\mathcal{O}_Y)$, which we denote by f_* .

We define the inverse image functor as follows:

$$f^*: \mathcal{C}(\mathcal{O}_X) \rightarrow \mathcal{C}(\mathcal{O}_Y): \mathcal{G} \mapsto f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Since f^* is right exact, we can define its left derived functor $D^-(\mathcal{O}_Y) \rightarrow D^-(\mathcal{O}_X)$, which we also denote by f^* . When we treat étale sheaves and \mathcal{O}_Y -modules at the same time, we write f^{-1} for the inverse image functor of étale sheaves to avoid confusion.

For any complex $\mathcal{K} \in D^-(\mathcal{O}_X)$, we write $\mathrm{adj}: \mathcal{K} \rightarrow f_*f^*\mathcal{K}$ for the adjunction map which coincides with the composite of the adjunction map $\mathcal{K} \rightarrow f_*f^{-1}\mathcal{K}$ defined in (2.1.1) and the canonical morphism $f_*f^{-1}\mathcal{K} \rightarrow f_*f^*\mathcal{K}$.

If X and Y are smooth over S and f is proper, then by [SGA6, Exp. III, Cor. 4.8.1], f_* sends $D_{\mathrm{perf}}(\mathcal{O}_X)$ to $D_{\mathrm{perf}}(\mathcal{O}_Y)$.

Let $g: Y \rightarrow X$ be a finite étale morphism. Then we have the trace map

$$\mathrm{Tr}_{g,\mathcal{O}_X}: g_*\mathcal{O}_Y \rightarrow \mathcal{O}_X.$$

Since g is étale and finite, for any complex \mathcal{K} in $D(\mathcal{O}_X)$, we have a canonical isomorphism $\mathcal{K} \otimes_{\mathcal{O}_X} g_*\mathcal{O}_Y \xrightarrow{\sim} g_*g^*\mathcal{K}$. Thus, by tensoring $\mathrm{Tr}_{g,\mathcal{O}_X}$ with \mathcal{K} and composing the canonical isomorphism, we obtain

$$\mathrm{Tr}_{g,\mathcal{K}}: g_*g^*\mathcal{K} \rightarrow \mathcal{K}.$$

This morphism is functorial with respect to \mathcal{K} .

2.1.3 Relation between \mathbb{Z}/p^n -modules and \mathcal{O}_X -modules

We suppose that X and Y are smooth over S . Let $f: Y \rightarrow X$ and $g: Y \rightarrow X$ be morphisms of schemes over S . We assume that f is proper and g is finite and étale.

Let K be a complex in $D(X, \mathbb{Z}/p^n)$ and \mathcal{K} a complex in $D^-(\mathcal{O}_X)$. Let $u: K \rightarrow \mathcal{K}$ be a morphism of complexes. We define the morphism

$$f_* f^* u: f_* f^{-1} K \rightarrow f_* f^* \mathcal{K}$$

by the composite of

$$f_* f^{-1} u: f_* f^{-1} K \rightarrow f_* f^{-1} \mathcal{K}$$

and the canonical map

$$f_* f^{-1} \mathcal{K} \rightarrow f_* f^* \mathcal{K}.$$

We also define $g_* g^* u$ by the same way.

Proposition 2.1. *The diagrams*

$$\begin{array}{ccc} K & \xrightarrow{\text{adj}} & f_* f^{-1} K \\ u \downarrow & & \downarrow f_* f^* u \\ \mathcal{K} & \xrightarrow{\text{adj}} & f_* f^* \mathcal{K} \end{array}$$

and

$$\begin{array}{ccc} g_* g^{-1} K & \xrightarrow{\text{Tr}} & K \\ g_* g^* u \downarrow & & \downarrow u \\ g_* g^* \mathcal{K} & \xrightarrow{\text{Tr}} & \mathcal{K} \end{array}$$

are commutative.

Proof. The commutativity of the first diagram follows from the definition of $f_* f^* u$ and $\text{adj}: \mathcal{K} \rightarrow f_* f^* \mathcal{K}$. For the commutativity of the second, it suffices to show the diagram

$$\begin{array}{ccc} g_* g^{-1} \mathcal{K} & \xrightarrow{\text{Tr}} & \mathcal{K} \\ \downarrow & \nearrow \text{Tr} & \\ g_* g^* \mathcal{K} & & \end{array}$$

is commutative. This follows from the commutativity of the diagrams

$$\begin{array}{ccc} \mathcal{K} \otimes_{\mathbb{Z}/p^n} g_* \mathbb{Z}/p^n & \xrightarrow{\sim} & g_* g^{-1} \mathcal{K} \\ \downarrow & & \downarrow \\ \mathcal{K} \otimes_{\mathcal{O}_X} g_* \mathcal{O}_Y & \xrightarrow{\sim} & g_* g^* \mathcal{K} \end{array}$$

and

$$\begin{array}{ccc} g_*\mathbb{Z}/p^n & \xrightarrow{\text{Tr}} & \mathbb{Z}/p^n \\ \downarrow & & \downarrow \\ g_*\mathcal{O}_Y & \xrightarrow{\text{Tr}} & \mathcal{O}_X. \end{array}$$

□

2.2 Cohomological correspondences

2.2.1 Correspondences

Let X and Γ be separated schemes of finite type over S . Let $a: \Gamma \rightarrow X \times_S X$ be a S -morphism of schemes. Such a morphism is called a correspondence. For a correspondence a , we denote by $\text{Fix } a$ the fiber product $\Gamma \times_{X \times_S X} X$, that is, $\text{Fix } a$ is given by the cartesian diagram

$$\begin{array}{ccc} \text{Fix } a & \longrightarrow & X \\ \downarrow & & \downarrow \Delta_{X/S} \\ \Gamma & \xrightarrow{a} & X \times_S X, \end{array}$$

where the morphism $\Delta_{X/S}$ is the diagonal map. We write a_1 (resp. a_2) for the composite of a and the first (resp. second) projection of $X \times_S X$.

In this section, we assume that a_1 is proper and a_2 is étale.

Let K be a complex in $D(X, \Lambda)$. We define $u_!$ by the composition

$$R\Gamma_c(X, K) \xrightarrow{R\Gamma_c(\text{adj})} R\Gamma_c(\Gamma, a_1^*K) \xrightarrow{R\Gamma_c(u)} R\Gamma_c(\Gamma, a_2^*K) \xrightarrow{R\Gamma_c(\text{Tr})} R\Gamma_c(X, K).$$

When the schemes X and Γ are proper over S , we often denote $u_!$ by u_* .

For any geometric point z of $\text{Fix } a$, we put $K_z = (a_1^*K)_z = (a_2^*K)_z$. Then the stalk

$$u_z: K_z = (a_1^*K)_z \rightarrow (a_2^*K)_z = K_z$$

is an endomorphism on K_z . If K is a complex in $D_{\text{ctf}}^b(X, \Lambda)$ then K_z lies in $D_{\text{perf}}(\Lambda)$. Hence we can consider the trace $\text{Tr}(u_z | K_z)$, which is an element of Λ .

Now, we assume that X and Γ are proper over S . In this case, a_2 is finite.

For any complex \mathcal{K} in $D^-(\mathcal{O}_X)$ and $u \in \text{Hom}(a_1^*\mathcal{K}, a_2^*\mathcal{K})$, we can also define the endomorphism u_* on $R\Gamma(X, \mathcal{K})$ in the same way.

Let \mathcal{K} be a complex in $D^-(\mathcal{O}_X)$ and u a morphism in $\text{Hom}(a_1^*\mathcal{K}, a_2^*\mathcal{K})$. For any connected component β of $\text{Fix } a$, we put $\mathcal{K}_\beta = i_\beta^*a_1^*\mathcal{K} = i_\beta^*a_2^*\mathcal{K}$ and

$$u_\beta = i_\beta^*u: \mathcal{K}_\beta = i_\beta^*a_1^*\mathcal{K} \rightarrow i_\beta^*a_2^*\mathcal{K} = \mathcal{K}_\beta,$$

where we write i_β for the closed immersion $\beta \hookrightarrow \Gamma$.

We assume that X , Γ and $\text{Fix } a$ are smooth over S . If \mathcal{K} is a complex in $D_{\text{perf}}(\mathcal{O}_X)$, by [SGA6, Exp. III, Cor. 4.5.1, Rem. 4.6.2 and Exp. VII, Prop. 1.9], we have $\mathcal{K}_\beta = i_\beta^*a_2^*\mathcal{K}_\beta \in D_{\text{perf}}(\mathcal{O}_\beta)$. So we can consider the trace $\text{Tr}(u_\beta | \mathcal{K}_\beta)$, which is an element of $\Gamma(\beta, \mathcal{O}_\beta)$.

2.2.2 Push-forward for correspondences

Let X and Γ be separated S -schemes of finite type. We consider open immersions $j_X: X \hookrightarrow \overline{X}$ and $j_\Gamma: \Gamma \hookrightarrow \overline{\Gamma}$, and correspondences $a: \Gamma \rightarrow X$ and $\overline{a}: \overline{\Gamma} \rightarrow \overline{X}$ such that the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{a} & X \times_S X \\ j_\Gamma \downarrow & & \downarrow j_X \times j_X \\ \overline{\Gamma} & \xrightarrow{\overline{a}} & \overline{X} \times_S \overline{X} \end{array}$$

is commutative. We assume that a_1 and \overline{a}_1 are proper. Let K be a complex in $D(X, \Lambda)$. Then we define the morphism

$$BC_2: j_{\Gamma!} a_2^* K \rightarrow \overline{a}_2^* j_{X!} K$$

by the composition

$$j_{\Gamma!} a_2^* K \xrightarrow{j_{\Gamma!} a_2^*(\text{adj})} j_{\Gamma!} a_2^* j_X^* j_{X!} K = j_{\Gamma!} j_\Gamma^* \overline{a}_2^* j_{X!} K \xrightarrow{\text{adj}} \overline{a}_2^* j_{X!} K.$$

By the properness of a_1 and \overline{a}_1 , we also have the morphism

$$j_{X!} K \xrightarrow{j_{X!}(\text{adj})} j_{X!} a_{1*} a_1^* K = j_{X!} a_{1!} a_1^* K = \overline{a}_{1!} j_{\Gamma!} a_1^* K = \overline{a}_{1*} j_{\Gamma!} a_1^* K.$$

Thus, by adjointness, we can define the morphism

$$BC_1: \overline{a}_1^* j_{X!} K \rightarrow j_{\Gamma!} a_1^* K.$$

For any $u \in \text{Hom}(a_1^* K, a_2^* K)$, we define $j_! u \in \text{Hom}(\overline{a}_1^* j_{X!} K, \overline{a}_2^* j_{X!} K)$ by the composition

$$\overline{a}_1^* j_{X!} K \xrightarrow{BC_1} j_{\Gamma!} a_1^* K \xrightarrow{j_{\Gamma!} u} j_{\Gamma!} a_2^* K \xrightarrow{BC_2} \overline{a}_2^* j_{X!} K.$$

2.2.3 Frobenius correspondences

Let X_0 be a \mathbb{F}_q -scheme and G_0 a sheaf on X_0 . Then we have an canonical isomorphism

$$\text{Fr}_{G_0}: \text{Fr}_{X_0}^* G_0 \xrightarrow{\sim} G_0.$$

We call it the Frobenius correspondence of G_0 (see [SGA4 $\frac{1}{2}$, Rapport, 1.2]). The Frobenius correspondence is functorial, that is, for any morphism $f_0: X_0 \rightarrow Y_0$ of \mathbb{F}_q -schemes and any sheaf G_0 on Y_0 , the pull-back $f_0^* \text{Fr}_{G_0}$ is the Frobenius correspondence of $f_0^* G_0$. For any integer $m \geq 1$, we put

$$\text{Fr}_{G_0}^m = \text{Fr}_{G_0} \circ \text{Fr}_{X_0}^* \text{Fr}_{G_0} \circ \cdots \circ \text{Fr}_{X_0}^{(m-1)*} \text{Fr}_{G_0}.$$

We define an isomorphism $\mathrm{Fr}_G: \mathrm{Fr}_X^* G \xrightarrow{\sim} G$ as the pull-back of Fr_{G_0} by $X \rightarrow X_0$ and also call it the Frobenius correspondence of G . Note that Fr_G is functorial as well as Fr_{G_0} . We define Fr_G^m similarly to $\mathrm{Fr}_{G_0}^m$.

By abuse of notation, we often write Fr for Fr_{G_0} or Fr_G .

Since Fr is isomorphic, we can replace the morphism u_0 in Theorem 1.2 by the morphism $u_0 \circ \mathrm{Fr}^m$ where $u_0 \in \mathrm{Hom}(a_{01}^* G_0, a_{02}^* G_0)$. When we will state our result later, we use the replaced notation.

3 The case of an automorphism of finite order

First, we recall the Lefschetz trace formula for the Frobenius correspondence which is proved in [SGA4 $\frac{1}{2}$, Fonction L mod. ℓ]. After that, we extend this formula to the case of an automorphism of finite order by using the method in the proof of [DL, Proposition 3.3].

Theorem 3.1 ([SGA4 $\frac{1}{2}$, Fonction L mod. ℓ , Thm. 4.1]). *Let X_0 be a separated scheme of finite type over \mathbb{F}_q , and A a noetherian reduced ring of characteristic p . Then for any object K_0 in $D_{\mathrm{ctf}}^b(X_0, A)$ and any integer $m \geq 1$, we have*

$$\mathrm{Tr}((\mathrm{Fr}_K^m)_! \mid R\Gamma_c(X, K)) = \sum_{x \in \mathrm{Fix}(\mathrm{Fr}_X^m \times \mathrm{id})} \mathrm{Tr}((\mathrm{Fr}_K^m)_x \mid K_x).$$

Corollary 3.2. *Let X_0 be a separated scheme of finite type over \mathbb{F}_q and g_0 an automorphism of order r on X_0 . Then for any constructible \mathbb{F}_p -module G_0 on X_0 , any integer $m \geq 1$ and any morphism $u_0 \in \mathrm{Hom}(g_0^* G_0, G_0)$ of order r , we have*

$$\mathrm{Tr}((u \circ \mathrm{Fr}^m)_! \mid R\Gamma_c(X, G)) = \sum_{x \in \mathrm{Fix}(g^{(m)} \times \mathrm{id})} \mathrm{Tr}((u \circ \mathrm{Fr}^m)_x \mid G_x).$$

Proof. By changing the field of definition of X , we may assume that $m = 1$.

Since X_0 is separated of finite type over \mathbb{F}_q and g_0 is of finite order, we can construct a finite partition X_{0i} of X_0 such that X_{0i} is locally closed in X_0 , affine, g_0 -stable for each i , and G_0 is smooth on X_{0i} . Then we have

$$\mathrm{Tr}((u \circ \mathrm{Fr})_! \mid R\Gamma_c(X, G)) = \sum_i \mathrm{Tr}((u \circ \mathrm{Fr})_! \mid R\Gamma_c(X_i, G_i))$$

and

$$\sum_{x \in \mathrm{Fix}(g^{(1)} \times \mathrm{id})} \mathrm{Tr}((u \circ \mathrm{Fr})_x \mid G_x) = \sum_i \sum_{x \in \mathrm{Fix}(g_i^{(1)} \times \mathrm{id})} \mathrm{Tr}((u \circ \mathrm{Fr})_x \mid G_x),$$

where g_i (resp. G_i) is the restriction of g (resp. G) to X_i . Hence we may assume that X_0 is affine and G_0 is smooth.

We put $X_1 = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$, $G_1 = G_0|_{X_1}$ and $u_1 = u_0|_{X_1}$. We denote by σ the isomorphism on the scheme $\mathrm{Spec} \mathbb{F}_{q^r}$ associated to the q -th power map on

\mathbb{F}_{q^r} . Since X_1 is affine, by using $g_0 \otimes \sigma^{-1}$ as a descent datum, we can construct the scheme X'_0 over \mathbb{F}_q such that $X'_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r} \simeq X_1$ and $g^{(1)}$ is the relative Frobenius endomorphism of X with respect to X_0 . Since G_1 is smooth, we can also construct the sheaf G'_0 on X'_0 such that $G'_0|_{X_1} \simeq G_1$ and $u \circ \text{Fr}$ is equal to the Frobenius correspondence on G with respect to G'_0 . Therefore by Theorem 3.1 we obtain

$$\text{Tr}((u \circ \text{Fr})_! \mid R\Gamma_c(X, G)) = \sum_{x \in \text{Fix}(g^{(1)} \times \text{id})} \text{Tr}((u \circ \text{Fr})_x \mid G_x).$$

□

4 Woods Hole formula

In this section we prove a trace formula for coherent sheaves called “Woods Hole formula”. We use this theorem in Section 6 and 7.

Theorem 4.1. *Let S be a spectrum of an artinian local ring. Let X and Γ be proper smooth schemes over S . We denote by π_X the structure morphism of X . Let $a: \Gamma \hookrightarrow X \times_S X$ be a closed immersion over S . Assume that a_2 is étale and the homomorphism $da_1: a_1^* \Omega_{X/S} \rightarrow \Omega_{\Gamma/S}$ induced by a_1 is zero. Then for any object \mathcal{K} in $D_{\text{perf}}(\mathcal{O}_X)$ and $u \in \text{Hom}(a_1^* \mathcal{K}, a_2^* \mathcal{K})$, we have*

$$\text{Tr}(u_* \mid R\Gamma(X, \mathcal{K})) = \sum_{\beta \in \pi_0(\text{Fix } a)} \text{Tr}_{\beta/S}(\text{Tr}(u_\beta \mid \mathcal{K}_\beta)).$$

Remark 4.2. • Since X and Γ are smooth over S and $da_1 = 0$, Γ meets transversally to the diagonal of X , that is, $\text{Fix } a$ is étale over S [EGA, IV Cor. 17.13.6]. Hence $\text{Fix } a$ is a finite direct sum of the spectrums of local artinian rings which are étale over S .

- Assume that $S = \text{Spec } W_n(k)$ and X is defined over $W_n(\mathbb{F}_q)$. We suppose that there exists a lift $\text{Fr}_X: X \rightarrow X$ of the relative Frobenius endomorphism on $X \otimes_{W_n(k)} k$. Then $d\text{Fr}_X^n = 0$. Thus the condition $da_1 = 0$ holds if we twist the given correspondence by the n -th power of the Frobenius lift Fr_X .

Before starting the proof, we recall a notation of residue symbols. We consider the following commutative diagram

$$\begin{array}{ccc} \beta & \xrightarrow{i} & Z \\ & \searrow g & \downarrow f \\ & & S, \end{array}$$

where i is a closed immersion, f is smooth of relative dimension d , and g is étale and finite.

Since g is étale, the natural morphism

$$\mathcal{I}_\beta/\mathcal{I}_\beta^2 \rightarrow \Omega_{Z/S}^1 \otimes_{\mathcal{O}_Z} \mathcal{O}_Z/\mathcal{I}_\beta$$

is isomorphic. Putting \bigwedge^d to the both sides of the above isomorphism, we have

$$\bigwedge^d \mathcal{I}_\beta/\mathcal{I}_\beta^2 \xrightarrow{\sim} \Omega_{Z/S}^d \otimes_{\mathcal{O}_Z} \mathcal{O}_Z/\mathcal{I}_\beta.$$

We note that $\bigwedge^d \mathcal{I}_\beta/\mathcal{I}_\beta^2$ is an invertible sheaf on β .

Now we take an \mathcal{O}_Z -sequence s_1, \dots, s_d generating the defining ideal \mathcal{I}_β of β and a global section ω of $\Omega_{Z/S}^d$. We denote by $\bar{\omega}$ the global section of $\Omega_{Z/S}^d \otimes_{\mathcal{O}_Z} \mathcal{O}_Z/\mathcal{I}_\beta$ obtained from ω and by $\bar{s}_1, \dots, \bar{s}_d$ the global sections of $\mathcal{I}_\beta/\mathcal{I}_\beta^2$ obtained from s_1, \dots, s_d . By the above isomorphism, we can regard $\bar{\omega}$ as a global section of $\bigwedge^d \mathcal{I}_\beta/\mathcal{I}_\beta^2$. For any $\tau \in \Gamma(\beta, \mathcal{O}_\beta)$, we define

$$\text{Res}_{Z/S} \begin{pmatrix} \tau & \omega & & \\ & s_1 & \dots & s_d \end{pmatrix} := \text{Tr}_{\beta/S} (\tau \cdot \bar{\omega} \otimes (d\bar{s}_1 \wedge \dots \wedge d\bar{s}_d)^{-1}),$$

which is an element of $\Gamma(S, \mathcal{O}_S)$.

Proof of Theorem 4.1. We put $m = \dim X$. We denote by d_1 (resp. d_2) the composite of $d : \mathcal{O}_{X \times_S X} \rightarrow \Omega_{X \times_S X/S}$ and the first (resp. second) projection of $\Omega_{X \times_S X/S} \xrightarrow{\sim} \text{pr}_2^* \Omega_{X/S} \oplus \text{pr}_1^* \Omega_{X/S}$.

By the corollary of Lefschetz-Verdier trace formula [SGA5, Exp. III, Thm. 6.10 and Rem. 6.11], we obtain

$$\begin{aligned} & \text{Tr}(u_* \mid R\Gamma(X, \mathcal{K})) \\ &= \sum_{\beta \in \pi_0(\text{Fix } a)} \text{Res}_{X \times_S X/S} \begin{pmatrix} \text{Tr}(u_\beta \mid \mathcal{K}_\beta) & d_1 s_1 \wedge \dots \wedge d_1 s_m \wedge d_2 t_1 \wedge \dots \wedge d_2 t_m \\ & s_1 & \dots & s_m & t_1 & \dots & t_m \end{pmatrix}, \end{aligned}$$

where s_1, \dots, s_m is an $\mathcal{O}_{X \times_S X}$ -sequence generating the defining ideal of Γ nearby β , and t_1, \dots, t_m is an $\mathcal{O}_{X \times_S X}$ -sequence defined as follows.

Since $a_2 \circ i_\beta$ is a closed immersion and β is étale over S , there are an open neighborhood X' of β in X and an étale morphism $X' \rightarrow S[T_1, \dots, T_m]$ such that the following diagram is commutative:

$$\begin{array}{ccc} \beta & \xrightarrow{\quad} & X' \\ \downarrow & & \downarrow \\ S & \xrightarrow{\quad} & S[T_1, \dots, T_m], \end{array}$$

where the bottom row is the zero section. We denote by x_1, \dots, x_m the sections of $\mathcal{O}_{X'}$ corresponding to T_1, \dots, T_m . By the above diagram, we obtain $a_2(x_i) \in \mathcal{I}_\beta$ for each i . In addition, $1 \otimes x_i - x_i \otimes 1, \dots, 1 \otimes x_m - x_m \otimes 1$ generate the

defining ideal of diagonal $\Delta_X: X \hookrightarrow X \times_S X$ nearby β since X' is étale over $S[T_1, \dots, T_m]$. We put $t_i = 1 \otimes x_i - x_i \otimes 1$.

Then we obtain

$$\begin{aligned}
& \text{Res} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{cccccc} d_1 s_1 \wedge \cdots \wedge d_1 s_m \wedge d_2 t_1 \wedge \cdots \wedge d_2 t_m \\ s_1 & \cdots & s_m & t_1 & \cdots & t_m \end{array} \right) \\
&= \text{Res} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{cccccc} d_1 s_1 \wedge \cdots \wedge d_1 s_m \wedge d(1 \otimes x_1) \wedge \cdots \wedge d(1 \otimes x_m) \\ s_1 & \cdots & s_m & t_1 & \cdots & t_m \end{array} \right) \\
&= \text{Res} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{cccccc} ds_1 \wedge \cdots \wedge ds_m \wedge d(1 \otimes x_1) \wedge \cdots \wedge d(1 \otimes x_m) \\ s_1 & \cdots & s_m & t_1 & \cdots & t_m \end{array} \right) \\
&= \text{Res} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{cccccc} d(a_2(x_1)) \wedge \cdots \wedge d(a_2(x_m)) \\ a_2(x_1) - a_1(x_1) & \cdots & a_2(x_m) - a_1(x_m) \end{array} \right),
\end{aligned}$$

where we write Res instead of $\text{Res}_{X \times_S X/S}$ for simplicity. At the last equality we use the property of [Ha, III.9 (R3)].

By the definition of x_1, \dots, x_m , for each i we can write

$$a_2(x_i) = \sum_{j=1}^m c_{ij}(a_2(x_j) - a_1(x_j))$$

nearby β , where each c_{ij} is a section of \mathcal{O}_Γ . Since $a_1 \circ i_\beta = a_2 \circ i_\beta$ and $da_1 = 0$ by the assumption, we have

$$da_2(x_i) = \sum_{j=1}^m c_{ij} da_2(x_j)$$

as sections of $i_\beta^* \Omega_{\Gamma/S}$. Thus we obtain $c_{ij} = \delta_{ij}$ in \mathcal{O}_β , where δ_{ij} is Kronecker's delta. Therefore, by [Ha, III.9 (R6)], we have

$$\begin{aligned}
& \text{Res} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{cccccc} d(a_2(x_1)) \wedge \cdots \wedge d(a_2(x_m)) \\ a_2(x_1) - a_1(x_1) & \cdots & a_2(x_m) - a_1(x_m) \end{array} \right) \\
&= \text{Res} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{cccccc} \det(c_{ij}) \cdot d(a_2(x_1) - a_1(x_1)) \wedge \cdots \wedge d(a_2(x_m) - a_1(x_m)) \\ a_2(x_1) - a_1(x_1) & \cdots & a_2(x_m) - a_1(x_m) \end{array} \right) \\
&= \text{Tr}_{\beta/S}(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \det(c_{ij})|_\beta) \\
&= \text{Tr}_{\beta/S}(\text{Tr}(u_\beta \mid \mathcal{K}_\beta)).
\end{aligned}$$

□

5 Some linear algebra

In this section, we recall some linear algebra for Section 6 and 7. We write $\sigma_0: W_n(\mathbb{F}_q) \rightarrow W_n(\mathbb{F}_q)$ (resp. $\sigma: W_n(k) \rightarrow W_n(k)$) for an endomorphism induced by the p -th power map on \mathbb{F}_q (resp. on k). We put $d = [\mathbb{F}_q : \mathbb{F}_p]$.

Lemma 5.1. *Let M_0 be a $W_n(\mathbb{F}_q)$ -module of finite type and $\Phi_0: M_0 \rightarrow M_0$ a σ_0 -linear endomorphism. We put $F_0 = \Phi_0^d$ and $M = M_0 \otimes_{W_n(\mathbb{F}_q)} W_n(k)$. We write $\Phi: M \rightarrow M$ (resp. $F: M \rightarrow M$) for the σ -linear (resp. $W_n(k)$ -linear) extension of Φ_0 (resp. F_0) to M . Then the following assertions hold:*

- (1) *The morphism $1 - \Phi: M \rightarrow M$ is surjective.*
- (2) *We have $M^{1-\Phi}/p(M^{1-\Phi}) = (M/pM)^{1-\Phi}$.*
- (3) *The canonical morphism $M^{1-\Phi} \otimes_{\mathbb{Z}/p^n} W_n(k) \rightarrow M$ is injective. Moreover, Φ and F is nilpotent on the cokernel of this morphism.*

Proof. (1) If $n = 1$, this assertion follows from [SGA7, Exp. XXII, Prop. 1.2].

We assume that the assertion holds for $n - 1$. We have the commutative diagram of the exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & pM & \longrightarrow & M & \longrightarrow & M/pM \longrightarrow 0 \\ & & \downarrow 1-\Phi & & \downarrow 1-\Phi & & \downarrow 1-\Phi \\ 0 & \longrightarrow & pM & \longrightarrow & M & \longrightarrow & M/pM \longrightarrow 0. \end{array}$$

We note that pM (resp. M/pM) has a natural $W_{n-1}(k)$ -module (resp. k -vector space) structure of finite type. By the assumption of the induction (resp. the case $n = 1$), the endomorphism $1 - \Phi$ on pM (resp. M/pM) is surjective. Hence, by the above exact sequence, $1 - \Phi$ on M is surjective.

(2) Applying the snake lemma to the above diagram, we have the exact sequence

$$0 \rightarrow (pM)^{1-\Phi} \rightarrow M^{1-\Phi} \rightarrow (M/pM)^{1-\Phi} \rightarrow 0.$$

Hence we have $M^{1-\Phi}/(pM)^{1-\Phi} = (M/pM)^{1-\Phi}$.

On the other hand, we have the commutative diagram of the exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & M[p] & \longrightarrow & M & \xrightarrow{\times p} & pM \longrightarrow 0 \\ & & \downarrow 1-\Phi & & \downarrow 1-\Phi & & \downarrow 1-\Phi \\ 0 & \longrightarrow & M[p] & \longrightarrow & M & \xrightarrow{\times p} & pM \longrightarrow 0, \end{array}$$

where we put $M[p] = \text{Ker}(M \xrightarrow{\times p} M)$. We note that $M[p]$ has a natural k -vector space structure of finite type. As we have shown above, the endomorphism $1 - \Phi$ on $M[p]$ is surjective. Applying the snake lemma to this diagram, we have $p(M^{1-\Phi}) = (pM)^{1-\Phi}$.

Therefore we obtain $M^{1-\Phi}/p(M^{1-\Phi}) = (M/pM)^{1-\Phi}$.

(3) If $n = 1$, the assertion follows from [SGA7, Exp. XXII, Cor. 1.1.10]. We assume that the assertion holds for $n - 1$.

By the proved assertion (2), we have the exact sequence

$$0 \longrightarrow (pM)^{1-\Phi} \longrightarrow M^{1-\Phi} \longrightarrow (M/pM)^{1-\Phi} \longrightarrow 0.$$

Tensoring this diagram with $W_n(k)$, we obtain the exact sequence

$$0 \rightarrow (pM)^{1-\Phi} \otimes W_{n-1}(k) \rightarrow M^{1-\Phi} \otimes W_n(k) \rightarrow (M/pM)^{1-\Phi} \otimes k \rightarrow 0,$$

where we write

$$\otimes W_{n-1}(k), \quad \otimes W_n(k) \quad \text{and} \quad \otimes k$$

for

$$\otimes_{\mathbb{Z}/p^{n-1}} W_{n-1}(k), \quad \otimes_{\mathbb{Z}/p^n} W_n(k) \quad \text{and} \quad \otimes_{\mathbb{F}_p} k.$$

We have the commutative diagram of the exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & (pM)^{1-\Phi} \otimes W_{n-1}(k) & \rightarrow & M^{1-\Phi} \otimes W_n(k) & \rightarrow & (M/pM)^{1-\Phi} \otimes k \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & pM & \longrightarrow & M & \longrightarrow & M/pM \longrightarrow 0. \end{array}$$

By the assumption of the induction (resp. the case $n = 1$), the left (resp. right) vertical arrow of the above diagram is injective. Hence the morphism

$$M^{1-\Phi} \otimes W_n(k) \rightarrow M$$

is injective.

We prove the latter statement. We put

$$M' = \text{Coker}(M^{1-\Phi} \otimes W_n(k) \rightarrow M).$$

Applying the snake lemma to the above diagram, we have

$$M'/pM' \simeq \text{Coker}((M/pM)^{1-\Phi} \otimes k \rightarrow M/pM).$$

We need the following lemma:

Lemma 5.2. *Let M' be a $W_n(k)$ -module and $\Phi: M' \rightarrow M'$ be a \mathbb{Z}/p^n -linear endomorphism. We assume that Φ is nilpotent on M'/pM' . Then Φ is nilpotent on M' .*

Proof. For the assumption, there exists an integer $e \geq 1$ such that $\Phi^e = 0$ on M'/pM' . Let x_0 be an element of M' . For each integer $i \geq 0$, we can inductively find $x_{i+1} \in M'$ such that $\Phi^e(x_i) = px_{i+1}$. Thus we have $\Phi^{ne}(x) = p^n x_n = 0$. Therefore Φ is nilpotent. \square

We return to the proof of Lemma 5.1.(3). By the case $n = 1$, the endomorphism Φ on M'/pM' is nilpotent. Therefore, by Lemma 5.2, Φ is also nilpotent on M' .

We will show that F is nilpotent on M' . We fix an integer e such that $\Phi^{de} = 0$ on M'/pM' . Let x_0 be an element of M_0 and α an element of $W_n(k)$.

We denote by $x_0 \otimes \alpha$ the class of $x_0 \otimes \alpha \in M$ in M' . By Lemma 5.2, it suffices to show that the element

$$F^e(x_0 \otimes \alpha) = \alpha(\Phi_0^{de}(x_0) \otimes 1)$$

belongs to pM' . By the definition of e , the element

$$\Phi^{de}(x_0 \otimes \alpha) = \sigma^{de}(\alpha)(\Phi_0^{de}(x_0) \otimes 1)$$

belongs to pM' . Hence we have either $\sigma^{de}(\alpha) \in pW_n(k)$ or $\Phi_0^{de}(x_0) \otimes 1 \in pW_n(k)$. For the former case, we have $\alpha \in pW_n(k)$. Therefore, in both cases, we have $F^e(x_0 \otimes \alpha) \in pM'$. \square

Lemma 5.3. *We use the notation of Lemma 5.1. We assume that M is free over $W_n(k)$ and $M^{1-\Phi}$ is free over \mathbb{Z}/p^n . Then there exists an integer $N \geq 1$ such that the following assertion holds: Let φ be a $W_n(k)$ -linear endomorphism on M which is defined over $W_n(\mathbb{F}_q)$ and satisfies $\varphi \circ \Phi = \Phi \circ \varphi$. For any $m \geq N$, we have*

$$\text{Tr}(\varphi \circ F^m \mid M^{1-\Phi}) = \text{Tr}(\varphi \circ F^m \mid M).$$

If $n = 1$, then $N = 1$ satisfies the above condition.

Remark 5.4. By the assumption of φ , we have $\varphi \circ F = F \circ \varphi$ and $M^{1-\Phi}$ is $\varphi \circ F^m$ -stable for any m .

Proof of Lemma 5.3. By Lemma 5.1(3), we have the exact sequence

$$0 \rightarrow M^{1-\Phi} \otimes_{\mathbb{Z}/p^n} W_n(k) \rightarrow M \rightarrow M' \rightarrow 0,$$

where we put $M' = \text{Coker}(M^{1-\Phi} \otimes_{\mathbb{Z}/p^n} W_n(k) \rightarrow M)$. By the assumption, $M^{1-\Phi} \otimes W_n(k)$ is free $W_n(k)$ -module of finite rank. Since $W_n(k)$ is a Gorenstein ring, $M^{1-\Phi} \otimes W_n(k)$ is an injective $W_n(k)$ -module. Hence the above sequence splits. Thus we have

$$\text{Tr}(\varphi \circ F^m \mid M) = \text{Tr}(\varphi \circ F^m \mid M^{1-\Phi}) + \text{Tr}(\varphi \circ F^m \mid M').$$

Let N be the minimal number of the integers N' such that the equation $\Phi^{dN'} = 0$ on M' holds. By Lemma 5.1(3), this value is well-defined and we have $F^m = 0$ on M' for any $m \geq N'$. Therefore we have $\text{Tr}(\varphi \circ F^m \mid M') = 0$ and the equation in Lemma 5.3 holds.

We assume that $n = 1$. By Remark 5.4, $\varphi \circ F$ is nilpotent on M' . Hence all of the eigenvalues of $\varphi \circ F$ are zero. Therefore, for any $m \geq 1$, we have $\text{Tr}(\varphi \circ F^m \mid M') = 0$. \square

6 Main theorem for the case $n = 1$

In this section we prove Theorem 1.2 under the condition (ii).

Theorem 6.1. Let \overline{X}_0 and $\overline{\Gamma}_0$ be proper smooth \mathbb{F}_q -schemes and X_0 an open \mathbb{F}_q -subscheme of \overline{X}_0 such that the complement $\overline{X} \setminus X$ is associated to a Cartier divisor. Let $\overline{a}_0: \overline{\Gamma}_0 \rightarrow \overline{X}_0 \times_{\mathbb{F}_q} \overline{X}_0$ be an \mathbb{F}_q -morphism. We write $a_0: \Gamma_0 \rightarrow X_0 \times_k X_0$ for the morphism such that the following diagram is cartesian:

$$\begin{array}{ccc} \Gamma_0 & \xrightarrow{a_0} & X_0 \times_{\mathbb{F}_q} X_0 \\ \downarrow & \square & \downarrow \\ \overline{\Gamma}_0 & \xrightarrow{\overline{a}_0} & \overline{X}_0 \times_{\mathbb{F}_q} \overline{X}_0. \end{array}$$

Suppose that a_1 is proper, \overline{a}_2 is étale, and \overline{a} is a closed immersion. Then, for any integer $m \geq 1$, any smooth constructible \mathbb{F}_p -sheaf \overline{G}_0 on X_0 , and any $u_0 \in \text{Hom}(a_{01}^* G_0, a_{02}^* G_0)$, we obtain

$$\text{Tr}((u \circ \text{Fr}^m)_! \mid R\Gamma_c(X, G)) = \sum_{z \in \text{Fix } a^{(m)}} \text{Tr}((u \circ \text{Fr}^m)_z \mid G_z),$$

where we put $G_0 = \overline{G}_0|_{X_0}$.

Remark 6.2. (a) Since a_{01} is proper, the following diagram is cartesian as that of topological spaces:

$$\begin{array}{ccc} \Gamma & \xrightarrow{a_{01}} & X \\ j_{\Gamma} \downarrow & & \downarrow j_X \\ \overline{\Gamma} & \xrightarrow{\overline{a}_{01}} & \overline{X}. \end{array}$$

Thus we have $\overline{a}_{02}^{-1}(\overline{X}_0 \setminus X_0) \subset \overline{a}_{01}^{-1}(\overline{X}_0 \setminus X_0)$ as topological spaces.

(b) Since \overline{G}_0 is smooth and X_0 is dense in \overline{X}_0 , there exists a unique morphism $\overline{u}_0: \overline{a}_{01}^* \overline{G}_0 \rightarrow \overline{a}_{02}^* \overline{G}_0$ such that $\overline{u}_0|_{\Gamma_0} = u_0$ (see for example [SGA4, Exp. XVI, Prop. 3.2]).

Proof of Theorem 6.1. By changing the field of definition of X , we may assume that $m = 1$.

We put $\overline{\mathcal{G}}_0 = \overline{G}_0 \otimes_{\mathbb{F}_p} \mathcal{O}_{\overline{X}_0}$ and $\mathcal{G}_0 = \overline{\mathcal{G}}_0|_{X_0} = G_0 \otimes_{\mathbb{F}_p} \mathcal{O}_{X_0}$. We denote by \mathcal{I}_0 the reduced defining ideal of $\overline{X}_0 \setminus X_0$. We put $\mathcal{G}'_0 = \mathcal{I}_0 \overline{\mathcal{G}}_0$, which is a locally free $\mathcal{O}_{\overline{X}_0}$ -module since \mathcal{I} is locally generated by a regular section. We note that we have $\mathcal{G}'_0 = \overline{G}_0 \otimes_{\mathbb{F}_p} \mathcal{I}_0$ since \overline{G}_0 is a smooth \mathbb{F}_q -sheaf.

If Y is a scheme of characteristic p , we define the morphism f_Y of schemes by $f_Y = (\text{id}_Y, \Phi_{\mathcal{O}_Y})$, where $\Phi_{\mathcal{O}_Y}: \mathcal{O}_Y \rightarrow \mathcal{O}_Y$ is the p -th power map. For a p -linear morphism of \mathcal{O}_Y -module sheaves $\Phi: \mathcal{G}_1 \rightarrow \mathcal{G}_2$, we denote by $\Phi': f_Y^* \mathcal{G}_1 \rightarrow \mathcal{G}_2$ the \mathcal{O}_Y -linear homomorphism associated to Φ .

We put $\Phi_{\overline{\mathcal{G}}_0} = \text{id}_{\overline{\mathcal{G}}_0} \otimes \Phi_{\mathcal{O}_{\overline{X}_0}}$ and we also define $\Phi_{\mathcal{G}_0}$, $\Phi_{\overline{\mathcal{G}}}$, and $\Phi_{\mathcal{G}}$ as well. By abuse of notation, we sometimes denote by Φ these p -linear homomorphisms of sheaves. We put $F_0 = \text{id}_{\overline{\mathcal{G}}_0} \otimes \Phi'_{\mathcal{O}_{\overline{X}_0}}: f_{X_0}^* \overline{\mathcal{G}}_0 \rightarrow \overline{\mathcal{G}}_0$.

Then we have

$$F_0(\mathcal{I}_0^n \overline{\mathcal{G}}_0) \subset \mathcal{I}_0^{np} \overline{\mathcal{G}}_0.$$

Thus the morphism $F_0^d: \mathrm{Fr}_{X_0}^* \overline{\mathcal{G}}_0 \rightarrow \overline{\mathcal{G}}_0$ is factorized as

$$\begin{array}{ccc} \mathrm{Fr}_{X_0}^* \mathcal{G}'_0 & \xrightarrow{\quad \succ \quad} & \mathcal{I}_0^q \overline{\mathcal{G}}_0 \hookrightarrow \mathcal{G}'_0 \\ \downarrow & & \downarrow \\ \mathrm{Fr}_{X_0}^* \overline{\mathcal{G}}_0 & \xrightarrow{F_0^d} & \overline{\mathcal{G}}_0, \end{array} \quad (1)$$

where we put $d = [\mathbb{F}_q : \mathbb{F}_p]$. Changing the base of the above diagram by $\mathbb{F}_q \rightarrow k$, we obtain the diagram

$$\begin{array}{ccc} \mathrm{Fr}_X^* \mathcal{G}' & \xrightarrow{\quad} & \mathcal{I}^q \overline{\mathcal{G}} \hookrightarrow \mathcal{G}' \\ \downarrow & & \downarrow \\ \mathrm{Fr}_X^* \overline{\mathcal{G}} & \xrightarrow{\quad} & \overline{\mathcal{G}}. \end{array} \quad (2)$$

We write $\mathrm{Fr}_{\overline{\mathcal{G}}}$ for the bottom horizontal morphism and $\mathrm{Fr}_{\mathcal{G}'}$ for the composition of the top horizontal morphisms. For simplicity, we sometimes write Fr for these morphisms and their pull-backs.

We have

$$(\overline{a}_{01}^{-1} \mathcal{I}_0 \cdot \mathcal{O}_{\overline{\Gamma}_0})_{\mathrm{red}} \subset (\overline{a}_{02}^{-1} \mathcal{I}_0 \cdot \mathcal{O}_{\overline{\Gamma}_0})_{\mathrm{red}}$$

by Remark 6.2 (a). Since \overline{a}_{02} is étale, we obtain

$$\overline{a}_{01}^{-1} \mathcal{I}_0 \cdot \mathcal{O}_{\overline{\Gamma}_0} \subset (\overline{a}_{01}^{-1} \mathcal{I}_0 \cdot \mathcal{O}_{\overline{\Gamma}_0})_{\mathrm{red}} \subset (\overline{a}_{02}^{-1} \mathcal{I}_0 \cdot \mathcal{O}_{\overline{\Gamma}_0})_{\mathrm{red}} = \overline{a}_{02}^{-1} \mathcal{I}_0 \cdot \mathcal{O}_{\overline{\Gamma}_0}.$$

Hence, by the flatness of \overline{a}_{02} , the morphism $\overline{u}_0 \otimes \mathrm{id}: \overline{a}_{01}^* \overline{\mathcal{G}} \rightarrow \overline{a}_{02}^* \overline{\mathcal{G}}$ is factorized as

$$\begin{array}{ccc} \overline{a}_{01}^* \mathcal{G}'_0 & \xrightarrow{u'_0} & \overline{a}_{02}^* \mathcal{G}'_0 \\ \downarrow & & \downarrow \\ \overline{a}_{01}^* \overline{\mathcal{G}}_0 & \xrightarrow{\overline{u}_0 \otimes \mathrm{id}} & \overline{a}_{02}^* \overline{\mathcal{G}}_0 \end{array} \quad (3)$$

and by the diagram (1) we have

$$(u'_0 \circ \mathrm{Fr})(\overline{a}_{01}^{(1)*} \mathcal{G}') \subset \overline{a}_{02}^* \mathcal{I}_0^q \overline{\mathcal{G}}_0 \subset \overline{a}_{02}^* \mathcal{I}_0 \mathcal{G}'_0. \quad (4)$$

Thus we have the diagram

$$\begin{array}{ccccccc} R\Gamma_c(X, G) & \xrightarrow{R\Gamma(\mathrm{adj})} & R\Gamma(\overline{\Gamma}, \overline{a}_1^* j_! G) & \xrightarrow{R\Gamma(j_! u)} & R\Gamma(\overline{\Gamma}, \overline{a}_2^* j_! G) & \xrightarrow{R\Gamma(\mathrm{Tr})} & R\Gamma_c(X, G) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R\Gamma(\overline{X}, \mathcal{G}') & \xrightarrow{R\Gamma(\mathrm{adj})} & R\Gamma(\overline{\Gamma}, \overline{a}_1^* \mathcal{G}') & \xrightarrow{R\Gamma(u')} & R\Gamma(\overline{\Gamma}, \overline{a}_2^* \mathcal{G}') & \xrightarrow{R\Gamma(\mathrm{Tr})} & R\Gamma(\overline{X}, \mathcal{G}'). \end{array} \quad (5)$$

We note that the composite of the top horizontal morphisms in the diagram (5) coincides with $u_!$ and that of the bottom horizontal morphisms is defined over \mathbb{F}_q .

We check the commutativity of the diagram (5). The commutativity of the left and right squares follows from Proposition 2.1. To show that the middle is commutative, it suffices to check the commutativity of the square

$$\begin{array}{ccc} \bar{a}_1^* j_! G & \xrightarrow{j_! u} & \bar{a}_2^* j_! G \\ \downarrow & & \downarrow \\ \bar{a}_1^* \mathcal{G}' & \xrightarrow{u'} & \bar{a}_2^* \mathcal{G}' \end{array} \quad (6)$$

In the following diagram

$$\begin{array}{ccccc} & & \bar{a}_1^* j_! G & \xrightarrow{\quad} & \bar{a}_2^* j_! G \\ & \swarrow & \downarrow & & \swarrow \downarrow \\ \bar{a}_1^* \bar{G} & \xrightarrow{\quad} & \bar{a}_2^* \bar{G} & & \\ \downarrow & & \downarrow & & \downarrow \\ & \swarrow & \bar{a}_1^* \mathcal{G}' & \xrightarrow{\quad} & \bar{a}_2^* \mathcal{G}' \\ & \swarrow & \downarrow & & \swarrow \downarrow \\ \bar{a}_1^* \bar{\mathcal{G}} & \xrightarrow{\quad} & \bar{a}_2^* \bar{\mathcal{G}} & & \end{array} \quad (*)$$

all the squares in the above diagram except (6) are commutative and the morphism (*) is injective. Thus the commutativity of the diagram (6) follows.

Also we have the commutative diagram

$$\begin{array}{ccccccc} R\Gamma(\bar{X}, \mathcal{G}') & \xrightarrow{R\Gamma(\text{adj})} & R\Gamma(\bar{\Gamma}, \bar{a}_1^* \mathcal{G}') & \xrightarrow{R\Gamma(u')} & R\Gamma(\bar{\Gamma}, \bar{a}_2^* \mathcal{G}') & \xrightarrow{R\Gamma(\text{Tr})} & R\Gamma(\bar{X}, \mathcal{G}') \\ R\Gamma(\Phi) \downarrow & & R\Gamma(\Phi) \downarrow & & R\Gamma(\Phi) \downarrow & & R\Gamma(\Phi) \downarrow \\ R\Gamma(\bar{X}, \mathcal{G}') & \xrightarrow{R\Gamma(\text{adj})} & R\Gamma(\bar{\Gamma}, \bar{a}_1^* \mathcal{G}') & \xrightarrow{R\Gamma(u')} & R\Gamma(\bar{\Gamma}, \bar{a}_2^* \mathcal{G}') & \xrightarrow{R\Gamma(\text{Tr})} & R\Gamma(\bar{X}, \mathcal{G}'). \end{array} \quad (7)$$

The commutativity of the right and left squares follows from the commutative squares

$$\begin{array}{ccc} f_{X_0}^* \mathcal{G}'_0 & \xrightarrow{\text{adj}} & \bar{a}_{01*} \bar{a}_{01}^* f_{X_0}^* \mathcal{G}'_0 \\ \Phi' \downarrow & & \Phi' \downarrow \\ \mathcal{G}'_0 & \xrightarrow{\text{adj}} & \bar{a}_{01*} \bar{a}_{01}^* \mathcal{G}'_0 \end{array}$$

and

$$\begin{array}{ccc} \bar{a}_{02*} \bar{a}_{02}^* f_{\bar{X}_0}^* \mathcal{G}'_0 & \xrightarrow{\text{Tr}} & f_{\bar{X}_0}^* \mathcal{G}'_0 \\ \Phi' \downarrow & & \Phi' \downarrow \\ \bar{a}_{02*} \bar{a}_{02}^* \mathcal{G}'_0 & \xrightarrow{\text{Tr}} & \mathcal{G}'_0. \end{array}$$

The commutativity of the middle of the diagram (7) follows from the commutative square

$$\begin{array}{ccc} \bar{a}_1^* \bar{\mathcal{G}} & \xrightarrow{\bar{u} \otimes \text{id}} & \bar{a}_2^* \bar{\mathcal{G}} \\ \Phi \downarrow & & \Phi \downarrow \\ \bar{a}_1^* \bar{\mathcal{G}} & \xrightarrow{\bar{u} \otimes \text{id}} & \bar{a}_2^* \bar{\mathcal{G}} \end{array}$$

and the commutative diagram (3).

By [SGA4 $\frac{1}{2}$, Function L mod. ℓ^n , Lem. 3.3], we obtain the exact sequence

$$0 \longrightarrow j_! G \longrightarrow \mathcal{G}' \xrightarrow{1-\Phi} \mathcal{G}' \longrightarrow 0. \quad (8)$$

which induces the following long exact sequence

$$\cdots \longrightarrow H_c^i(X, G) \longrightarrow H^i(\bar{X}, \mathcal{G}') \xrightarrow{1-H^i(\Phi)} H^i(\bar{X}, \mathcal{G}') \longrightarrow \cdots.$$

Since \bar{X} is proper over k and \mathcal{G}' is a coherent $\mathcal{O}_{\bar{X}}$ -module sheaf, the k -vector space $H^i(\bar{X}, \mathcal{G}')$ is finite-dimensional for each i . Hence by Lemma 5.1(1) the p -linear map $1 - H^i(\Phi)$ is surjective. Thus, for each index i , we have the exact sequence

$$0 \longrightarrow H_c^i(X, G) \longrightarrow H^i(\bar{X}, \mathcal{G}') \xrightarrow{1-H^i(\Phi)} H^i(\bar{X}, \mathcal{G}') \longrightarrow 0.$$

By the diagram (7), we have $u' \circ \Phi = \Phi \circ u'$. Therefore, by the diagram (5) and Lemma 5.3, we have

$$\begin{aligned} \text{Tr}((u \circ \text{Fr})_! \mid H_c^i(X, G)) &= \text{Tr}(u_! \circ (\text{Fr}_G)_! \mid H_c^i(X, G)) \\ &= \text{Tr}(u'_* \circ (\text{Fr}_{\mathcal{G}'})_* \mid H^i(\bar{X}, \mathcal{G}')) \\ &= \text{Tr}((u' \circ \text{Fr})_* \mid H^i(\bar{X}, \mathcal{G}')). \end{aligned} \quad (9)$$

On the other hand, since $d\text{Fr}_{\bar{X}} = 0$, \bar{a}_2 is étale and \mathcal{G}' is locally free of finite rank, we can apply Theorem 4.1 to $\text{Tr}((u' \circ \text{Fr})_* \mid R\Gamma(\bar{X}, \mathcal{G}'))$:

$$\text{Tr}((u' \circ \text{Fr})_* \mid R\Gamma(\bar{X}, \mathcal{G}')) = \sum_{z \in \text{Fix } \bar{a}^{(1)}} \text{Tr}((u' \circ \text{Fr})_z \mid \mathcal{G}'_z). \quad (10)$$

We note that, since in this case the base scheme is $\text{Spec } k$, we can identify connected components of $\text{Fix } \bar{a}^{(1)}$ with closed points of $\text{Fix } \bar{a}^{(1)}$ by associating β to the underlying space z of β .

- If z lies in $\text{Fix } a^{(1)}$, then we have

$$\text{Tr}((u' \circ \text{Fr})_z \mid \mathcal{G}'_z) = \text{Tr}((u \circ \text{Fr})_z \mid \mathcal{G}_z)$$

by the definition of u' .

- If z does not belong to $\text{Fix } a^{(1)}$, then $\text{Fr}_{\bar{\Gamma}} \circ \bar{a}_1(z) = \bar{a}_2(z) \in \bar{X} \setminus X$ by Remark 6.2(a). By the inclusion (4), $u' \circ \text{Fr}$ is factorized as follows:

$$\begin{array}{ccc} & & \bar{a}_2^* \mathcal{I} \mathcal{G}' \\ & \nearrow & \downarrow (** \\ \bar{a}_1^{(1)*} \mathcal{G}' & \xrightarrow{u' \circ \text{Fr}} & \bar{a}_2^* \mathcal{G}' \end{array}$$

In the above diagram, the pull-back of $(**)$ by i_z is zero. Thus we have $(u' \circ \text{Fr})_z = 0$ and $\text{Tr}((u' \circ \text{Fr})_z \mid \mathcal{G}'_z) = 0$.

Therefore, from (9), (10) and the above calculation of the local terms, we obtain Theorem 6.1. \square

7 Main theorem for the proper case

In this section we prove Theorem 1.2 under the condition (iii). We write σ_0 (resp. σ) for the Frobenius map on $W_n(\mathbb{F}_q)$ (resp. $W_n(k)$).

Theorem 7.1. *Let X and Γ be proper smooth k -schemes defined over \mathbb{F}_q . Let $a: \Gamma \hookrightarrow X \times X$ be a closed immersion defined over \mathbb{F}_q such that a_2 is étale. Let K_0 be a complex in $D_{\text{perf}}(X_0, \mathbb{Z}/p^n)$. We assume the following conditions:*

- *There exists a triplet of lifts $(\mathcal{X}_0, \tilde{\Gamma}_0, \tilde{a}_0)$ of (X_0, Γ_0, a_0) to $W_n(\mathbb{F}_q)$ such that \mathcal{X} and $\tilde{\Gamma}$ are smooth over $W_n(k)$, \tilde{a} is closed immersion, and \tilde{a}_2 is étale.*
- *There exists a σ_0 -linear endomorphism $\Phi_{\mathcal{X}_0}: \mathcal{O}_{\mathcal{X}_0} \rightarrow \mathcal{O}_{\mathcal{X}_0}$ which is a lift of the p -th power map $\Phi_{X_0}: \mathcal{O}_{X_0} \rightarrow \mathcal{O}_{X_0}$.*
- *$H^i(X, K)$ (resp. $H^i(\mathcal{X}, K \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{\mathcal{X}})$) is free over \mathbb{Z}/p^n (resp. $W_n(k)$) for each i .*

Then there exists an integer N such that, for any $m \geq N$ and any $u_0 \in \text{Hom}(a_{01}^ K_0, a_{02}^* K_0)$, we obtain*

$$\text{Tr}((u \circ \text{Fr}^m)_* \mid R\Gamma(X, K)) = \sum_{z \in \text{Fix } a^{(m)}} \text{Tr}((u \circ \text{Fr}^m)_z \mid K_z).$$

Proof. We put $\mathcal{S} = \text{Spec } W_n(k)$. Since the morphism $\text{Spec } k \rightarrow \mathcal{S}$ is surjective, radiciel and finite, the pull-back functor from the étale topos of \mathcal{X} (resp. $\tilde{\Gamma}$) to the étale topos of X (resp. Γ) is an equivalence of categories and $D(\mathcal{S}, \mathbb{Z}/p^n)$

is naturally identified with $D(\mathbb{Z}/p^n)$. Let \tilde{K}_0 be the complex on \mathcal{X}_0 such that $\tilde{K}_0|_{X_0} = K_0$ and $\tilde{u}_0: \tilde{a}_{01}^* \tilde{K}_0 \rightarrow \tilde{a}_{02}^* \tilde{K}_0$ the morphism such that $\tilde{u}_0|_{\Gamma_0} = u_0$. Then we have

$$\mathrm{Tr}((u \circ \mathrm{Fr}^m)_* \mid R\Gamma(X, K)) = \mathrm{Tr}((\tilde{u} \circ \mathrm{Fr}^m)_* \mid R\Gamma(\mathcal{X}, \tilde{K}))$$

and

$$\mathrm{Tr}((u \circ \mathrm{Fr}^m)_z \mid K_z) = \mathrm{Tr}((\tilde{u} \circ \mathrm{Fr}^m)_{\tilde{z}} \mid \tilde{K}_{\tilde{z}}),$$

for any $z \in \mathrm{Fix} a^{(m)}$, where \tilde{z} is the element of $\mathrm{Fix} \tilde{a}^{(m)}$ corresponding to z . Hence it suffices to show the formula of Theorem 7.1 for $(\mathcal{X}, \tilde{\Gamma}, \tilde{a})$.

For simplicity, we replace the symbols $(\mathcal{S}, \mathcal{X}, \tilde{\Gamma}, \tilde{a}, \tilde{u})$ with (S, X, Γ, a, u) .

By the existence of Φ_{X_0} , we have the short exact sequence

$$0 \longrightarrow \mathbb{Z}/p^n \longrightarrow \mathcal{O}_X \xrightarrow{1-\Phi} \mathcal{O}_X \longrightarrow 0.$$

We put $\mathcal{K} = K \otimes_{\mathbb{Z}/p^n} \mathcal{O}_X$. Tensoring this diagram with K over \mathbb{Z}/p^n and acting $R\Gamma(X, -)$, we have the distinguished triangle

$$R\Gamma(X, K) \longrightarrow R\Gamma(X, \mathcal{K}) \xrightarrow{1-\Phi} R\Gamma(X, \mathcal{K}) \longrightarrow R\Gamma(X, K)[1].$$

By Lemma 5.1 (1), we have the short exact sequence

$$0 \longrightarrow H^i(X, K) \longrightarrow H^i(X, \mathcal{K}) \xrightarrow{1-H^i(\Phi)} H^i(X, \mathcal{K}) \longrightarrow 0.$$

We put $f_{X_0} = (\mathrm{id}, \Phi_{X_0}): (X_0, \mathcal{O}_{X_0}) \rightarrow (X_0, \mathcal{O}_{X_0})$. Let $\Phi': f_{X_0}^* \mathcal{O}_{X_0} \rightarrow \mathcal{O}_{X_0}$ be the \mathcal{O}_{X_0} -linear homomorphism associated to Φ_{X_0} . We write $\mathrm{Fr}_{\mathcal{O}_X}$ for the pull back of Φ'^d by $W_n(\mathbb{F}_q) \rightarrow W_n(k)$ and define $\mathrm{Fr}_{\mathcal{K}} = \mathrm{Fr}_K \otimes \mathrm{Fr}_{\mathcal{O}_X}$. We put $u' = u \otimes \mathrm{id}_{\mathcal{O}_\Gamma}: a_1^* \mathcal{K} \rightarrow a_2^* \mathcal{K}$. By Lemma 5.3, we can take an integer N such that for any $m \geq N$ we have

$$\begin{aligned} \mathrm{Tr}((u \circ \mathrm{Fr}^m)_* \mid H^i(X, K)) &= \mathrm{Tr}(u_* \circ (\mathrm{Fr}_K)^m_* \mid H^i(X, K)) \\ &= \mathrm{Tr}(u'_* \circ (\mathrm{Fr}_{\mathcal{K}})^m_* \mid H^i(X, \mathcal{K})) \\ &= \mathrm{Tr}((u' \circ \mathrm{Fr}^m)_* \mid H^i(X, \mathcal{K})). \end{aligned}$$

Taking the alternating sum and applying Theorem 4.1, we obtain

$$\begin{aligned} \mathrm{Tr}((u \circ \mathrm{Fr}^m)_* \mid R\Gamma(X, K)) &= \mathrm{Tr}((u' \circ \mathrm{Fr}^m)_* \mid R\Gamma(X, \mathcal{K})) \\ &= \sum_{\beta \in \pi_0(\mathrm{Fix} a^{(m)})} \mathrm{Tr}((u' \circ \mathrm{Fr}^m)_\beta \mid \mathcal{K}_\beta) \\ &= \sum_{z \in \mathrm{Fix} a^{(m)}} \mathrm{Tr}((u \circ \mathrm{Fr}^m)_z \mid K_z). \end{aligned}$$

Therefore Theorem 7.1 holds. \square

8 Examples

We will see three examples. At first, we see an easy example that we can show Theorem 1.2 by direct calculation. Secondly, we apply this theorem to elliptic curves over a finite field. Finally, we see that this theorem would be false if coefficients are p -adic.

Example 8.1. We see an example of the affine line over a finite field. In the notation of Theorem 6.1, we put

- $m = 1$,
- $X_0 = \mathbb{A}_{\mathbb{F}_q}^1, \overline{X}_0 = \mathbb{P}_{\mathbb{F}_q}^1$,
- $a_0 = (g_0, \text{id}): \mathbb{A}_{\mathbb{F}_q}^1 \rightarrow \mathbb{A}_{\mathbb{F}_q}^1 \times_{\mathbb{F}_q} \mathbb{A}_{\mathbb{F}_q}^1$, where $g_0: \mathbb{A}_{\mathbb{F}_q}^1 \rightarrow \mathbb{A}_{\mathbb{F}_q}^1$ is defined by $x \mapsto x + 1$,
- $\overline{a}_0 = (\overline{g}_0, \text{id})$, where $\overline{g}_0: \mathbb{P}_{\mathbb{F}_q}^1 \rightarrow \mathbb{P}_{\mathbb{F}_q}^1$ is a natural extension of g_0 ,
- $\overline{G}_0 = \mathbb{Z}/p, \overline{u}_0 = \text{id}$.

We have $H_c^i(\mathbb{A}_k^1, \mathbb{Z}/p) = 0$ for each i . Hence the left-hand side of the formula in Theorem 6.1 is zero.

On the other hand, the right-hand side coincides with the number of fixed points of $g \circ \text{Fr}$, that is, the number of roots of the equation $x^q + 1 = x$ in k . It exactly equals q , hence it is zero modulo p .

Example 8.2. We show an easy application of Theorem 6.1. Let E_0 be an elliptic curve over \mathbb{F}_q and O the origin of E_0 . Let g_0 be the isogeny on E_0 . In the notation of Theorem 6.1, we put

- $m = 1$,
- $\overline{X}_0 = E_0, X_0 = E_0 \setminus \{O\}$,
- $\Gamma_0 = g_0^{-1}(X_0), j_0: \Gamma_0 \hookrightarrow X_0, g'_0 = g_0|_{\Gamma_0}$,
- $\overline{a}_0 = (g_0, \text{id}), a_0 = (g'_0, j_0)$,
- $\overline{G}_0 = \mathbb{Z}/p, \overline{u}_0 = \text{id}$.

Then the above data satisfy the condition of Theorem 6.1. The cohomology groups with proper supports of X are as follows:

$$H_c^i(X, \mathbb{Z}/p) = \begin{cases} 0 & i = 0 \\ H^i(E, \mathbb{Z}/p) & \text{otherwise.} \end{cases}$$

By using the Artin-Schreier sequence, we can compute $H^i(E, \mathbb{Z}/p) = 0$ for any integer $i \geq 2$. On the other hand, we can show that

$$\text{Fix } a^{(1)} = \text{Fix}(\text{Fr}_E \circ g) \setminus \{O\}.$$

Thus, by applying Theorem 6.1, we have

$$- \operatorname{Tr}((\operatorname{Fr}_E \circ g)^* | H^1(E, \mathbb{Z}/p)) \equiv \# \operatorname{Fix}(\operatorname{Fr}_E \circ g) - 1 \pmod{p}.$$

We fix a prime number $\ell \neq p$. Using the Lefschetz trace formula for ℓ -adic étale cohomology, we can show that

$$- \operatorname{Tr}((\operatorname{Fr}_E \circ g)^* | H^1(E, \mathbb{Q}_\ell)) \equiv \# \operatorname{Fix}(\operatorname{Fr}_E \circ g) - 1 \pmod{p}.$$

Note that $\operatorname{Tr}((\operatorname{Fr}_E \circ g)^* | H^1(E, \mathbb{Q}_\ell))$ is an integer. Thus we obtain

$$\operatorname{Tr}((\operatorname{Fr}_E \circ g)^* | H^1(E, \mathbb{Z}/p)) \equiv \operatorname{Tr}((\operatorname{Fr}_E \circ g)^* | H^1(E, \mathbb{Q}_\ell)) \pmod{p}.$$

If E_0 is supersingular, the Artin-Schreier sequence shows that $H^1(E, \mathbb{Z}/p) = 0$. Thus we have

$$\# \operatorname{Fix}(\operatorname{Fr}_E \circ g) \equiv 1 \pmod{p}$$

and

$$\operatorname{Tr}((\operatorname{Fr}_E \circ g)^* | H^1(E, \mathbb{Q}_\ell)) \equiv 0 \pmod{p}.$$

Example 8.3. We remark that Deligne's conjecture seems to be false if coefficients are p -adic. For example, in Example 8.1, we take $\overline{G}_0 = \mathbb{Z}_p$. We note that

$$H_c^i(\mathbb{A}_k^1, \mathbb{Z}/p^n) = 0$$

for each n and each i . We define $H_c^i(\mathbb{A}_k^1, \mathbb{Z}_p) := \varprojlim_n H_c^i(\mathbb{A}_k^1, \mathbb{Z}/p^n)$. Then we have $H_c^i(\mathbb{A}_k^1, \mathbb{Z}_p) = 0$. Hence the equation

$$\sum_i (-1)^i \operatorname{Tr}((\operatorname{Fr}^m)^* | H_c^i(\mathbb{A}_k^1, \mathbb{Z}_p)) = 0$$

holds for each integer $m \geq 1$. However, we have

$$\#\{x \in \mathbb{A}_k^1 \mid x^{q^m} + 1 = x\} = q^m,$$

which is not zero in \mathbb{Z}_p .

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